

EXACT VALUE OF TAMMES PROBLEM FOR  $N = 10$ 

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ABSTRACT. Let  $C_i$  ( $i = 1, \dots, N$ ) be the  $i$ -th open spherical cap of angular radius  $r$  and let  $M_i$  be its center under the condition that none of the spherical caps contains the center of another one in its interior. We consider the upper bound,  $r_N$ , (not the lower bound!) of  $r$  of the case in which the whole spherical surface of a unit sphere is completely covered with  $N$  congruent open spherical caps under the condition, sequentially for  $i = 2, \dots, N - 1$ , that  $M_i$  is set on the perimeter of  $C_{i-1}$ , and that each area of the set  $(\cup_{\nu=1}^{i-1} C_\nu) \cap C_i$  becomes maximum. In this paper, for  $N = 10$ , we found out that the solutions of our sequential covering and the solutions of the Tammes problem were strictly correspondent. Especially, we succeeded to obtain the exact value  $r_{10}$  for  $N = 10$ .

## 1. INTRODUCTION

The circle on the surface of a sphere is called a spherical cap. Among the problems of packing on the spherical surface, the closest packing of congruent spherical caps is the most famous, and is usually known as the Tammes problem [7]. The details of Tammes problem are as follows: “How must  $N$  congruent non-overlapping spherical caps be packed on the surface of a unit sphere so that the angular diameter of spherical caps will be as great as possible?” The Tammes problem is mathematically proved solutions were known for  $N = 1, \dots, 12$ , and  $24^1$  [1, 2, 8]. The systematic method of attaining these solutions has not been given. The exact values in the cases for  $N = 1, \dots, 9, 11, 12$  and  $24$  are known, but only in the case for  $N = 10$ , the value is approximate range  $[1.154479, 1.154480]$  by Danzer [1].

We considered the packing problem by a systematic method which is different from the approach by Danzer. As a result, we obtained the exact value of angular diameter of spherical caps in the Tammes problem for  $N = 10$  as following [6].

$$r_{10} = \tan^{-1} \left( \left( \frac{4}{\sqrt{3}} \cos \left( \frac{1}{3} \tan^{-1} \left( \frac{\sqrt{3}\sqrt{229}}{9} \right) \right) + 3 \right)^{\frac{1}{2}} \right) \quad (1)$$

$$\approx 1.1544798334192707378319618404230 \dots \quad \text{rad}.$$

## 2. OUTLINE OF METHOD

Let us show our systematic method. Suppose we have  $N$  congruent open spherical caps with angular radius  $r$  on the surface  $\mathcal{S}$  of the unit sphere and suppose that these spherical caps cover the whole spherical surface without any gap under the condition

<sup>1</sup>In recent years, Oleg Musin and Alexey Tarasov proved the solution of the Tammes problem for  $N = 13$  and  $14$  [3, 4].

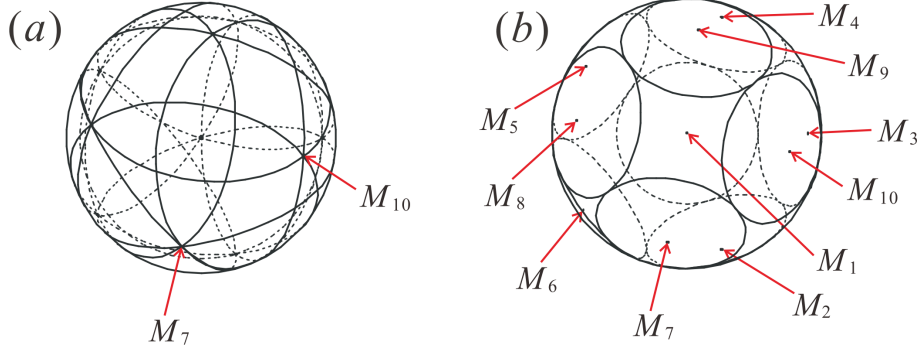


FIGURE. 1. (a) Our sequential covering for  $N = 10$ . (b) Our solution of Tammes problem for  $N = 10$ . Both viewpoints are  $(0, 0, 10)$ . In this example,  $M_1 = (0, 0, -1)$ ,  $M_2 \approx (0.26335, -0.87585, -0.40439)$ ,  $M_3 \approx (0.91458, 0, -0.40439)$ ,  $M_4 \approx (0.26335, 0.87585, -0.40439)$ ,  $M_5 \approx (-0.76292, 0.50440, -0.40439)$ ,  $M_6 \approx (-0.77575, -0.57681, -0.25593)$ ,  $M_7 \approx (-0.13883, -0.78326, 0.60599)$ ,  $M_8 \approx (-0.79006, 0.092588, 0.60599)$ ,  $M_9 \approx (0.084546, 0.74290, 0.66405)$ , and  $M_{10} \approx (0.735778, -0.13295, 0.66405)$ .

that none of them contains the center of another one in its interior. Further we suppose that the spherical caps are put on  $\mathcal{S}$  sequentially in the manner which is described just below. Let  $C_i$  be the  $i$ -th open spherical cap and let  $M_i$  be its center ( $i = 1, \dots, N$ ). Our problem is to calculate the upper bound of  $r$  for the sequential covering, such that  $N$  congruent open spherical caps cover the whole spherical surface  $\mathcal{S}$  under the condition that  $M_i$  is set on the perimeter of  $C_{i-1}$ , and that each area of set  $(\cup_{\nu=1}^{i-1} C_\nu) \cap C_i$  becomes maximum in sequence for  $i = 2, \dots, N-1$ . Here, we define a *half-cap* as the spherical cap whose angular radius is  $\frac{\pi}{2}$  and which is concentric with that of the original cap. Let us suppose the centers of  $N$  half-caps are placed on the positions of the centers of spherical caps  $C_i$  ( $i = 1, \dots, N$ ) which are considered. At this time, we get the packing with  $N$  congruent half-caps. Therefore, our sequential covering is in connection with the packing problem [5, 6].

We calculated the upper bound for  $N = 2, \dots, 12$  in our problem theoretically; the case  $N = 1$  is self-evident. As a result, we found the interesting fact that the solutions of our problem are strictly correspondent to those of the Tammes problem for  $N = 2, \dots, 12$  [5, 6]. Especially, as mentioned above, we succeeded to obtain the exact value for  $N = 10$  (see (1)) [6]. In the case for  $N = 10$ , when the centers are put on the spherical surface  $\mathcal{S}$  according to our method, their arrangements are as shown in Figure 1, for example. Hereafter, let us explain simply how to calculate the value of (1). When the centers of two spherical caps with angular radius  $r$  are put respectively at  $(0, 0, -1)$  and  $(\sin r, 0, -\cos r)$ , we can obtain the coordinates  $(x, y, z)$  of cross points where the perimeters of their spherical caps intersect by using simultaneous equations as follows:

$$\begin{cases} -z = \cos r, \\ \sin r \cdot x - \cos r \cdot z = \cos r, \\ x^2 + y^2 + z^2 = 1, \end{cases} \quad (2)$$

Solving (2), we obtain

$$\left( -\frac{\cos r (\cos r - 1)}{\sin r}, \frac{(\cos r - 1) \sqrt{2 \cos r + 1}}{\sin r}, -\cos r \right), \quad (3)$$

$$\left( -\frac{\cos r (\cos r - 1)}{\sin r}, -\frac{(\cos r - 1) \sqrt{2 \cos r + 1}}{\sin r}, -\cos r \right). \quad (4)$$

In our method, taking into account Theorem 1 in [5], the centers  $M_1 = (x_{m1}, y_{m1}, z_{m1})$ ,  $M_2 = (x_{m2}, y_{m2}, z_{m2})$ , and  $M_3 = (x_{m3}, y_{m3}, z_{m3})$  are set at  $(0, 0, -1)$ , the coordinates of (3), and  $(\sin r, 0, -\cos r)$ , respectively. Here, let  $\partial C_i$  be the perimeter of  $C_i$  ( $i = 1, \dots, 10$ ). Next, according to our method, we choose  $M_4 = (x_{m4}, y_{m4}, z_{m4})$  on the coordinates of (4). Then,  $M_5 = (x_{m5}, y_{m5}, z_{m5})$  is put on one of the cross points of  $\partial C_4$  and  $\partial C_1$ , and let it be outside  $C_3$  (i.e.,  $(\sin r (\cos^2 r - 2 \cos r - 1) / (\cos r + 1)^2, 2 \cos r \sin r \sqrt{2 \cos r + 1} / (\cos r + 1)^2, -\cos r)$ ). In addition, we put  $M_6 = (x_{m6}, y_{m6}, z_{m6})$  on one of the cross points of  $\partial C_5$  and  $\partial C_2$ , and let it be outside  $C_1$  (i.e.,  $(2 \cos r \sin r (\cos r - 1) (2 \cos r + 1) / (9 \cos^3 r - \cos^2 r - \cos r + 1), 2 \cos r \sin r (\cos r - 1) \sqrt{2 \cos r + 1} / (9 \cos^3 r - \cos^2 r - \cos r + 1), (-4 \cos^4 r + \cos^3 r - 5 \cos^2 r - \cos r + 1) / (9 \cos^3 r - \cos^2 r - \cos r + 1))$ ). At the time  $C_6$  is put on a spherical surface  $\mathcal{S}$ , the uncovered region is reduced to a pentagon which is bounded by perimeters of spherical caps. Then, in order to put the centers of four spherical caps on the pentagonal uncovered region, we consider a spherical square of side-length  $r$  on the pentagon. It is because, when the centers  $M_7, M_8, M_9$ , and  $M_{10}$  are put on the vertices of spherical square of side-length  $r$ , the set  $\cup_{\nu=1}^{10} C_\nu$  can cover the whole of  $\mathcal{S}$  with our method. Now, in our study, the centers  $M_7, M_8, M_9$ , and  $M_{10}$  are put respectively on the cross points of  $\partial C_6$  and  $\partial C_2$ ,  $\partial C_7$  and  $\partial C_6$ ,  $\partial C_8$  and  $\partial C_4$ , and  $\partial C_9$  and  $\partial C_3$ . Note that the cross points chosen as arrangement of  $M_7, M_8, M_9$ , and  $M_{10}$  are on the boundary of the pentagonal uncovered region. Therefore, from the coordinates of  $M_2$  and  $M_6$ , the coordinates of the centers  $M_7 = (x_{m7}, y_{m7}, z_{m7})$  are as obtained follows:

$$\begin{aligned} & \left( \frac{\sin r (\cos^3 r - 5 \cos^2 r - \cos r + 1)}{9 \cos^3 r - \cos^2 r - \cos r + 1}, -\frac{4 \sin r \cos^2 r \sqrt{2 \cos r + 1}}{9 \cos^3 r - \cos^2 r - \cos r + 1}, \right. \\ & \left. -\frac{\cos r (\cos^3 r + 11 \cos^2 r - \cos r - 3)}{9 \cos^3 r - \cos^2 r - \cos r + 1} \right). \end{aligned}$$

Next, we calculate the coordinates of  $M_{10}$  without using the coordinates of  $M_7$ . Here, let  $s_{i,j}$  denote the spherical distance between  $M_i$  and  $M_j$ . Then, by applying the spherical cosine theorem to the spherical isosceles triangle  $M_6 M_7 M_{10}$  of legs  $s_{6,7} = s_{7,10} = r$ , we have

$$\cos(s_{6,10}) = \frac{3 \cos^3 r + 2 \cos^2 r - \cos r - 2 (1 - \cos^2 r) \sqrt{\cos r + 2 \cos^2 r}}{(1 + \cos r)^2}.$$

It is because the inner angle at  $M_6$  of the spherical isosceles triangle  $M_6M_7M_{10}$  is the sum of the interior angles of spherical equilateral triangle  $M_6M_7M_8$  and spherical square  $M_7M_{10}M_9M_8$ . In this connection, the inner angle of spherical equilateral triangle of side-length  $r$  is  $\cos^{-1}(\cos r/(\cos r + 1))$ , and the inner angle of spherical square of side-length  $r$  is  $\cos^{-1}((\cos r - 1)/(\cos r + 1))$ . As a result, we can obtain the coordinates of  $M_{10} = (x_{m10}, y_{m10}, z_{m10})$  as a function of  $r$  through the simultaneous equations as follows:

$$\begin{cases} x_{m3} \cdot x_{m10} + y_{m3} \cdot y_{m10} + z_{m3} \cdot z_{m10} = \cos r, \\ x_{m6} \cdot x_{m10} + y_{m6} \cdot y_{m10} + z_{m6} \cdot z_{m10} = \frac{3 \cos^3 r + 2 \cos^2 r - \cos r - 2(1 - \cos^2 r) \sqrt{\cos r + 2 \cos^2 r}}{(1 + \cos r)^2}, \\ x_{m10}^2 + y_{m10}^2 + z_{m10}^2 = 1. \end{cases}$$

Note that, in this report, the coordinates of  $M_{10}$  are omitted since they are too complicated. Further, we get the equation of the following type

$$x_{m7} \cdot x_{m10} + y_{m7} \cdot y_{m10} + z_{m7} \cdot z_{m10} = \cos r. \quad (5)$$

When the equation (5) is solved against  $r$  by using mathematical software, we obtained the value of (1).

Danzer have solved the packing problem for  $N = 10$  through the consideration on irreducible graphs obtained by connecting those points, among  $N$  points, whose spherical distance is exactly the minimal distance [1]. Then he needed the independent considerations for  $N = 10$ . On the other hand, our systematic method is able to obtain a solution for  $N$  by using the results for the case  $N - 1$  or  $N - 2$ . In addition, we have considered the packing problem from the standpoint of sequential covering. The advantages of our approach are that we only need to observe uncovered region in the process of packing and that this uncovered region decreases step by step as the packing proceeds.

By using our method, the solutions of Tammes problem were obtained for  $N = 2, \dots, 12$  [5, 6]. This fact is interesting and it is important that the exact value for  $N = 10$  is found [6].

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#### REFERENCES

- [1] L. Danzer, *Endliche punktmengen auf der 2-sphäre mit möglichst großen Minimalabstand*. Universität Göttingen, 1963. (Finite point-set on  $S^2$  with minimum distance as large as possible, *Discrete Mathematics*, **60** (1986) 3–66.)
- [2] L. Fejes Tóth, *Lagerungen in der Ebene, auf der Kugel und im Raum*. Second edition. Springer-Verlag, Heidelberg, 1972. (Japanese trans. by I. Higuchi, M. Tanemura, *Haichi no mondai – heimen-kyuumen-kuukan ni okeru*. Misuzusyobou, Tokyo, 1983).
- [3] O.R. Musin, A.S. Tarasov, The strong thirteen spheres problem, *Discrete & Computational Geometry*, **48** (2012) 128–141. Available online: <http://dx.doi.org/10.1007/s00454-011-9392-2>, <http://arxiv.org/abs/1002.1439> (accessed on 9 September 2015).

- [4] —, The Tammes problem for  $N = 14$ , *Experimental Mathematics*, **24** (2015) 460–468. Available online: <http://www.tandfonline.com/doi/full/10.1080/10586458.2015.1022842>, <http://arxiv.org/abs/1410.2536> (accessed on 9 September 2015).
- [5] T. Sugimoto, M. Tanemura, Packing and Minkowski covering of congruent spherical caps on a sphere for  $N = 2, \dots, 9$ , *Forma*, **21** (2006) 197–225. Available online: <http://www.scipress.org/journals/forma/pdf/2103/21030197.pdf> (accessed on 30 August 2015).
- [6] T. Sugimoto, M. Tanemura, Packing and Minkowski covering of congruent spherical caps on a sphere, II: Cases of  $N = 10, 11$  and  $12$ , *Forma*, **22** (2007) 157–175. Available online: <http://www.scipress.org/journals/forma/pdf/2202/22020157.pdf> (accessed on 30 August 2015).
- [7] P.M.L. Tammes, On the origin of number and arrangement of the places of exit on the surface of pollen grains, *Recueil de Travaux Botaniques Néerlandais*, **27** (1930) 1–84.
- [8] Y. Teshima, T. Ogawa, Dense packing of equal circle on a sphere by the minimum-zenith method: Symmetrical Arrangement, *Forma*, **15** (2000) 347–364. Available online: <http://www.scipress.org/journals/forma/pdf/1504/15040347.pdf> (accessed on 30 August 2015).

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